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Pure spinor superfields, with application to D=3 conformal models

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Abstract: I review and discuss the construction of supersymmetry multiplets and manifestly supersymmetric Batalin–Vilkovisky actions using pure spinors, with emphasis on models with maximal supersymmetry. The special cases of $D = 3$, $N = 8$ (Bagger–Lambert–Gustavsson) and $N = 6$ (Aharony–Bergman–Jafferis–Maldacena) conformal models are treated in detail. Most of the material is covered by the papers arXiv:0808.3242 and arXiv:0809.0318. This is the written version of a talk given at 4th Baltic-Nordic workshop “Algebra, Geometry and Mathematical Physics”, Tartu, Estonia, October 9–11, 2008, to appear in the Proceedings of the Estonian Academy of Sciences, vol 4, 2010.

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There is a close relation between supermultiplets and pure spinors. The algebra of covariant fermionic derivatives in flat superspace is generically of the form

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^c D_c = -2\gamma_{\alpha\beta}^c D_c . \quad (1)$$

If a bosonic spinor λ^α is *pure*, *i.e.*, if the vector part $(\lambda\gamma^a\lambda)$ of the spinor bilinear vanishes, the operator $Q = \lambda^\alpha D_\alpha$ becomes nilpotent, and may be used as a BRST operator. This is, schematically, the starting point for pure spinor superfields. (The details of course depend on the actual space-time and the amount of supersymmetry. The pure spinor constraint may need to be further specified. Eq. (1) may also contain more terms, due to super-torsion and curvature.) The cohomology of Q will consist of supermultiplets, which in case of maximal supersymmetry are on-shell. The idea of manifesting maximal supersymmetry off-shell by

using pure spinor superfields $\Psi(x, \theta, \lambda)$ is to find an action whose equations of motion is $Q\Psi = 0$.

The fact that pure spinors had a rôle to play in maximally supersymmetric models was recognised early by Nilsson [1] and Howe [2,3]. Pure spinor superfields were developed with the purpose of covariant quantisation of superstrings by Berkovits [4,5,6,7] and the cohomological structure was independently discovered in supersymmetric field theory and supergravity, originally in the context of higher-derivative deformations [8,9,10,11,12,13,14,15]. The present lecture only deals with pure spinors for maximally supersymmetric field theory.

The canonical example of pure spinors is in $D = 10$. There is only one non-gravitational supermultiplet, namely super-Yang-Mills, so this is what we expect to obtain. Expanding a field $\Psi(x, \theta, \lambda)$ in powers of λ , one has

$$\Psi(x, \theta, \lambda) = \sum_{n=0}^{\infty} \lambda^{\alpha_1} \dots \lambda^{\alpha_n} \psi_{\alpha_1 \dots \alpha_n}(x, \theta) . \quad (2)$$

The implementation of the pure spinor constraint is as an abelian gauge symmetry, where the generators $(\lambda \gamma^a \lambda)$ act multiplicatively. The field Ψ is defined modulo the ideal generated by the constraint. A "canonical" representative of the gauge orbits is provided by superfields $\psi_{\alpha_1 \dots \alpha_n}(x, \theta)$ which, in addition to being symmetric, are completely γ -traceless, *i.e.*, in the modules $(000n0)$ of the Lorentz algebra (where λ^α is in (00001) and D_α in (00010) , the two spinor chiralities).

In order to calculate the cohomology, we start by finding the cohomology of zero-modes, x -independent fields. This cohomology is easy to calculate (a purely algebraic calculation), and gives information about the full cohomology. It is worth noting that the zero-mode cohomology (which clearly would have been empty for an unconstrained λ) may be read off from the partition function for a pure spinor. It is in one-to-one correspondence (for a concrete explanation of this fact, using the reducibility of the pure spinor constraint, see the appendix of ref. [5] and ref. [16]) with the six terms in the nominator of the partition function

$$Z(t) = \frac{1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8}{(1-t)^{16}} = \frac{(1+t^2)(1+4t+t^2)}{(1-t)^{11}} . \quad (3)$$

(This partition function only counts the dimension of the space of monomials in λ with degree of homogeneity p as the coefficient of t^p . A more refined partitions function, specifying the actual Lorentz modules appearing, can of course be written down; for this I refer to ref. [16].) The zero-mode cohomology is illustrated in Table 1. There, each column represent a field in the expansion (2), and the vertical direction is the expansion in θ . The columns have been shifted so that components on the same row have the same dimension, *i.e.*, so that Q acts horizontally. Since λ carries ghost number 1 and dimension $-1/2$, the component field $\psi_{\alpha_1 \dots \alpha_n}$ has ghost number $\text{gh}(\Psi) - n$ and dimension $\dim(\Psi) + \frac{n}{2}$. It is natural to let

$\text{gh}(\Psi) = 1$ and $\dim(\Psi) = 0$ and take Ψ to be fermionic. Then the scalar (ghost number 1, dimension 0) in the first column is interpreted as the Yang–Mills ghost and the vector and spinor in the second column as the fields of the super–Yang–Mills multiplet (the field ψ_α of ghost number 0 and dimension $1/2$ is the lowest-dimensional connection component A_α in a superfield treatment of super–Yang–Mills). The remaining fields are the corresponding antifields, in the Batalin–Vilkovisky (BV) sense. It is striking that one inevitably is lead to the BV formalism. It of course exists also in a component formalism, but when one uses pure spinors it is not optional. This means that any action formed in this formalism will be a BV action, and that the appropriate consistency relation (encoding the generalised gauge symmetry) is the master equation.

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
$\dim = 0$	(00000)				
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

Table 1. The cohomology of the $D = 10$ SYM complex.

To go from the zero-mode cohomology to the complete cohomology, one easily convinces oneself that component fields in the modules contained in the zero-mode cohomology will be subject to differential constraints in the modules of the zero-mode cohomology in the next column to the right. This gives the proper relations for the linearised on-shell super–Yang–Mills multiplet. (If a multiplet is an off-shell representation of supersymmetry, as is generically the case for half-maximal or lower supersymmetry, there will consequently be no anti-fields in the cohomology. These instead come in a separate pure spinor superfield [12].)

This far, we have not considered the actual solutions of pure spinor constraint, but rather regarded the pure spinor as a book-keeping device. When one wants to write down an action, this is no longer possible. For an action, a measure is needed. The linearised action should be “ $\int \Psi Q \Psi$ ” for some suitable definition of “ \int ”. Clearly, “ \int ” must have ghost number -3 . In the cohomology, there is a singlet at $\lambda^3 \theta^5$. Defining a measure as a “residue”, picking the corresponding component, has the right ghost number, and also the

correct dimension. However, it is singular, so components of Ψ with high enough power in λ or θ drop out of the putative action defined in this manner, and the equation of motion $Q\Psi = 0$ does not follow. Still, the corresponding tensorial structure can be used for an invariant integral over λ . It is clear from the partition function (3) that λ contains 11 degrees of freedom (out of the 16 for an unconstrained spinor). Explicit solution of the pure spinor constraint also shows that when imposed on a *complex* spinor, only 5 out of the ten constraints are independent (see *e.g.* [5] for details). Defining the scalar at $\lambda^3\theta^5$ as $T_{\alpha_1\alpha_2\alpha_3}^{\beta_1\dots\beta_{11}}\varepsilon_{\beta_1\dots\beta_{16}}\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\beta_{12}}\theta^{\beta_{13}}\theta^{\beta_{14}}\theta^{\beta_{15}}\theta^{\beta_{16}}$, where T thus is a Lorentz invariant tensor, one defines the conjugate invariant tensor $\tilde{T}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\dots\beta_{11}}$, and the integration is

$$[d\lambda]\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3} \sim \tilde{T}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\dots\beta_{11}}d\lambda^{\alpha_1} \wedge \dots \wedge d\lambda^{\alpha_{11}} . \quad (4)$$

In ref. [7], Berkovits solved the problem how to make sense of this integration and using it as part of a non-singular measure for the pure spinor superspace. The solution involves a non-minimal set of pure spinor variables, which in addition to λ^α contains a bosonic conjugate spinor $\bar{\lambda}_\alpha$ (which in euclidean signature can be viewed as the complex conjugate of λ^α) obeying $(\bar{\lambda}\gamma^a\bar{\lambda}) = 0$ and a fermionic spinor r_α with $(\bar{\lambda}\gamma^a r) = 0$. The new BRST operator is $Q = \lambda^\alpha D_\alpha + \frac{\partial}{\partial \bar{\lambda}_\alpha} r_\alpha$, and its cohomology is independent of $\bar{\lambda}$ and r . One assigns ghost number -1 and dimension $1/2$ to $\bar{\lambda}$ and ghost number 0 and dimension $1/2$ to r . The measure for $\bar{\lambda}$ is the complex conjugate to the one defined in eq. (4) for λ , and for r :

$$[dr] \sim \star \tilde{T}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\dots\beta_{11}} \bar{\lambda}_{\alpha_1} \bar{\lambda}_{\alpha_2} \bar{\lambda}_{\alpha_3} \frac{\partial}{\partial r_{\beta_1}} \dots \frac{\partial}{\partial r_{\beta_{11}}} . \quad (5)$$

Using these integration measures, and the ordinary ones for x and θ , we list the dimensions and ghost numbers for the theory after dimensional reduction to D dimensions in Table 2. So, the ghost numbers match, and also the dimensions ($\frac{1}{g^2}$ has dimension $D - 4$ in D dimensions).

	gh#	dim
$d^D x$	0	$-D$
$d^{16}\theta$	0	8
$[d\lambda]$	8	-4
$[d\bar{\lambda}]$	-8	4
$[dr]$	-3	-4
total	-3	$-(D - 4)$

Table 2. The dimensions and ghost numbers of the $D = 10$ measure.

The λ and $\bar{\lambda}$ integrations are non-compact and need regularisation. In ref. [7] this is achieved, following ref. [17], by the insertion of a factor $N = e^{\{Q, \chi\}}$. Since this differs from 1 by a Q -exact term, the regularisation is independent of the choice of the fermion χ . The choice $\chi = -\bar{\lambda}_\alpha \theta^\alpha$ gives $N = e^{-\lambda^\alpha \bar{\lambda}_\alpha - r_\alpha \theta^\alpha}$ and regularises the bosonic integrations at infinity. At the same time, it explains how the term at θ^5 is picked out, this follows after integration over r . N has definite ghost number 0 for the assignments for ghost number and dimension above (although any other assignment gives the correct ghost number and dimension for the non- Q -exact part).

An action for ten-dimensional super-Yang-Mills (or any dimensional reduction) can now be written in the Chern-Simons-like form [5]

$$S = \frac{1}{2g^2} \int \langle \Psi, Q\Psi + \frac{1}{3}[\Psi, \Psi] \rangle_{\text{adj}} . \quad (6)$$

Note that there is no 4-point coupling. The component field 4-point coupling arises after elimination of unphysical components. One must however remember that this is a classical BV action. It obeys the classical master equation $(S, S) = 0$, where the anti-bracket takes the simple form

$$(A, B) = \int A \langle \frac{\overleftarrow{\delta}}{\delta \Psi(Z)} [dZ] \frac{\overrightarrow{\delta}}{\delta \Psi(Z)} \rangle_{\text{adj}} B . \quad (7)$$

In order to perform quantum calculations with path integral over Ψ , gauge fixing has to be implemented. This involves traditional gauge fixing (of the component gauge field) as well as elimination of the anti-fields. I will comment briefly on gauge fixing towards the end of the lecture.

As already mentioned, pure spinor formulations are relevant for BV action formulations of any maximally supersymmetric model (exceptions being models containing self-dual tensors). I would now like to illustrate how they may be used for 3-dimensional conformal models. The pure spinor actions turn out to have a much simpler structure than the component actions. There has recently been much interest in conformal three-dimensional theories. Following the discovery of the existence of a maximally supersymmetric ($N = 8$) interacting theory of scalar multiplets coupled to Chern-Simons, the Bagger-Lambert-Gustavsson (BLG) theory [18,19,20,21], much effort has been spent on trying to generalise the construction and to interpret it in terms of an AdS boundary model of multiple M2-branes. The interesting, but restrictive, algebraic structure of the model, containing a 3-algebra with antisymmetric structure constants, turned out to have only one finite-dimensional realisation [22,23], possible to interpret in term of two M2-branes [24,25] (see however refs. [26,27] dealing with the infinite-dimensional solution related to volume-preserving diffeomorphisms in three dimensions).

It then became an urgent question how the stringent requirements in the BLG theory could be relaxed. There are different possibilities. One may let the scalar product on the matter representation be degenerate [28]. This works at the level of equations of motion, but does not allow for an action principle. One may also go one step further, and add further null

directions to that degenerate case, which leads to scalar products with indefinite signature [29,30,31] (and consequently to matter kinetic terms with different signs). Or, finally, one may reduce the number of supersymmetries, specifically to $N = 6$, as proposed by Aharony, Bergman, Jafferis and Maldacena (ABJM) [32], or maybe even to lower N [33,34]. The $N = 6$ models were further studied in refs. [35,36,37,38,39,40,41] (among other papers). For recent developments in the theory of multiple membranes, we refer to ref. [42] and references given there. The literature on the subject is huge, and we apologise for omissions of references to relevant papers.

The superfield formulation of the BLG model was given in ref. [43] (see also ref. [44], where the on-shell superfields were constructed for the example of the BLG model based on the infinite-dimensional algebra of volume-preserving diffeomorphisms in three dimensions). A superfield formulation with $N = 1$ superfields was given in ref. [45] and with $N = 2$ superfields in ref. [46]. In ref. [43] we constructed an action in an $N = 8$ pure spinor superspace formulation of the BLG model, which covers all situations with $N = 8$ above except the ones with degenerate scalar product. The construction was essentially performed using minimal pure spinor variables, and the issue of the integration measure was more or less neglected (the measure was assumed to exist). In the subsequent paper [47] also the $N = 6$ ABJM models were treated, and integration measures were defined using non-minimal variables for both types of models.

Let us first briefly review the results of ref. [43]. Since the BLG model is maximally supersymmetric, component formulations and also usual superspace formulations are on-shell. There is no finite set of auxiliary fields. A pure spinor treatment is necessary in order to write an action in a generalised BRST setting.

The Lorentz algebra in $D = 3$ is $so(1, 2) \approx sl(2, \mathbb{R})$. The $N = 8$ theory has an $so(8)$ R-symmetry, and we choose the fermionic coordinates and derivatives to transform as $(\mathbf{2}, \mathbf{8}_s) = (1)(0010)$ under $sl(2) \oplus so(8)$. This representation is real and self-conjugate. The pure spinors transform in the same representation, and are written as $\lambda^{A\alpha}$, where A is the $sl(2)$ index and α the $so(8)$ spinor index. As usual, a BRST operator is formed as $Q = \lambda^{A\alpha} D_{A\alpha}$, D being the fermionic covariant derivative. The nilpotency of Q demands that

$$(\lambda^A \lambda^B) = 0, \quad (8)$$

where (\dots) denotes contraction of $so(8)$ spinor indices, since the superspace torsion has to be projected out. This turns out to be the full constraint*. As will soon be clear, it is essential that not only $(\lambda^A \sigma_{IJKL} \lambda^B)$ but also $\varepsilon_{AB} (\lambda^A \sigma_{IJ} \lambda^B)$ is left non-zero. These pure spinors are similar to those encountered in ref. [48].

* The vanishing of the "torsion representation" — the vector part of the spinor bilinear — is necessary, but does not always give the full pure spinor constraint. One example where further constraints are needed is $N = 4$, $D = 4$ super-Yang-Mills theory.

The “pure spinor wave function” for the Chern–Simons field is a fermionic scalar Ψ of (mass) dimension 0 and ghost number 1. For the matter multiplet we have a bosonic field Φ^I in the $so(8)$ vector representation (0)(1000) of dimension 1/2 and ghost number 0. In addition to the pure spinor constraint, the matter field is identified modulo transformations

$$\Phi^I \rightarrow \Phi^I + (\lambda^A \sigma^I \varrho_A) \quad (9)$$

for arbitrary ϱ . (This type of additional gauge invariance is typical for fields in some non-trivial module of the structure group. Without it, the cohomology would be the tensor product of the module with the cohomology of a field in the trivial module.)

In this minimal pure spinor formulation the fields are expanded in power series in λ , *i.e.*, in decreasing ghost number. The pure spinor partition function is easily calculated to be

$$Z_1(t) = \frac{1 - 3t^2 + 3t^4 - t^6}{(1 - t)^{16}} = \frac{(1 + t)^3}{(1 - t)^{13}}. \quad (10)$$

The partition for a matter field is

$$Z_8 = \frac{8 - 16t + 16t^3 - 8t^4}{(1 - t)^{16}} = 8 \frac{(1 + t)}{(1 - t)^{13}}. \quad (11)$$

These expressions seem to imply that the number of independent degrees of freedom of a pure spinor is 13, *i.e.*, that the pure spinor constraint in this case is irreducible. This is verified by a concrete solution of the constraint (for a complex λ) [47]. As for the $D = 10$ pure spinors, the partition functions can of course be refined to include not only number of fields, but also modules of the structure group.

The field content (ghosts, fields and their antifields) is read off from the zero-mode BRST cohomology given in Tables 3 and 4 for the Chern–Simons and matter sectors respectively.

gh# =	1	0	−1	−2	−3
dim = 0	(0)(0000)				
$\frac{1}{2}$	•	•			
1	•	(2)(0000)	•		
$\frac{3}{2}$	•	•	•	•	
2	•	•	(2)(0000)	•	•
$\frac{5}{2}$	•	•	•	•	•
3	•	•	•	(0)(0000)	•
$\frac{7}{2}$	•	•	•	•	•

Table 3. The cohomology of the scalar complex.

gh# =	0	-1	-2	-3	-4
dim = $\frac{1}{2}$	(0)(1000)				
1	(1)(0001)	•			
$\frac{3}{2}$	•	•	•		
2	•	(1)(0001)	•	•	
$\frac{5}{2}$	•	(0)(1000)	•	•	•
3	•	•	•	•	•
$\frac{7}{2}$	•	•	•	•	•

Table 4. The cohomology of the vector complex.

We observe that the field content is the right one. In Ψ we find the ghost, the gauge connection, its antifield and the antighost. The antifield has dimension 2 (as opposed to *e.g.* $D = 10$ super-Yang–Mills, where it has dimension 3), indicating equations of motion that are first order in derivatives. It is quite striking that the (bosonic) Chern–Simons model has a natural supersymmetric off-shell extension, although the supersymmetry becomes trivial on-shell. It is not meaningful to talk about a gaugino field. In Φ we find the eight scalars ϕ^I , the fermions $\chi^{A\dot{\alpha}}$ and their antifields. In addition, the field Ψ transforms in the adjoint representation **adj** of some gauge group and Φ^I in some representation **R** of the gauge group. The corresponding indices are suppressed.

In order to derive the equations of motion for the physical component fields, one starts from the ghost number zero part of the fields (*i.e.*, $\Phi^I \rightarrow \phi^I(x, \theta)$ and $\Psi \rightarrow \lambda^\alpha A_\alpha(x, \theta)$ respectively), and examines the content of the θ expansion by repeated application of fermionic covariant derivatives, using the pure spinor constraint and the reducibility (9) when they occur. As a guideline one has the cohomology at ghost number one; these representations are the only ones where an equation of motion may sit, for obvious reasons. In this manner, one derives the linearised component equations $\square\phi^I = 0$, $\not{\partial}\chi^A = 0$ for the scalar multiplet, and $dA = 0$ for the Chern–Simons field, and also the interacting equations from the actions below.

In ref. [43], it was assumed that a non-degenerate measure can be formed using a non-minimal extension of the pure spinor variables along the lines of ref. [7]. This measure, including the three-dimensional integration, should carry dimension 0 and ghost number -3 , and should allow “partial integration” of the BRST charge Q . It was then shown that the Lagrangian of the interacting model is of a very simple form, containing essentially a Chern–Simons like term for the Chern–Simons field, minimally coupled to the matter sector:

$$S = \int \langle \Psi, Q\Psi + \frac{1}{3}[\Psi, \Psi] \rangle_{\text{adj}} + \int \frac{1}{2} M_{IJ} \langle \Phi^I, Q\Phi^J + \Psi \cdot \Phi^J \rangle_{\mathbf{R}} . \quad (12)$$

The brackets denote (non-degenerate) scalar products on \mathbf{adj} and \mathbf{R} , $[\cdot, \cdot]$ the Lie bracket of the gauge algebra and $T \cdot x$ the action of the Lie algebra element in the representation \mathbf{R} . M_{IJ} is the pure spinor bilinear $\varepsilon_{AB}(\lambda^A \sigma_{IJ} \lambda^B)$, which is needed for several reasons: in order to contract the indices on the Φ 's antisymmetrically, to get a Lagrangian of ghost number 3, and to ensure invariance in the equivalence classes defined by eq. (9).

The invariances of the interacting theory (equivalent to the classical master equation $(S, S) = 0$), generalising the BRST invariance in the linearised case, are:

$$\begin{aligned}\delta\Psi &= Q\Psi - [\Lambda, \Psi] - M_{IJ}\{\Phi^I, \Xi^J\} , \\ \delta\Phi^I &= -\Lambda \cdot \Phi^I + (Q + \Psi \cdot)\Xi^I ,\end{aligned}\tag{13}$$

where Λ is an adjoint boson of dimension 0 and ghost number 0, and Ξ^I a fermionic vector in \mathbf{R} of dimension 1/2 and ghost number -1 . Here we also introduced the bracket $\{\cdot, \cdot\}$ for the formation of an adjoint from the antisymmetric product of two elements in \mathbf{R} , defined via $\langle x, T \cdot y \rangle_{\mathbf{R}} = \langle T, \{x, y\} \rangle_{\mathbf{adj}}$. The invariance with parameter Λ is manifest. The transformation with Ξ has to be checked. One then finds that the transformation of the matter field Φ gives a “field strength” contribution from the anticommutator of the two factors $Q + \Psi$, which is cancelled against the variation of the Chern–Simons term. The single remaining term comes from the transformation of the Ψ in the covariant matter kinetic term, and it is proportional to $M_{IJ}M_{KL}\langle\{\Phi^I, \Phi^J\}, \{\Phi^K, \Xi^L\}\rangle_{\mathbf{adj}}$. Due to the pure spinor constraint, $M_{[IJ}M_{KL]} = 0$. This was shown in ref. [43], using the simple observation that the only $sl(2)$ singlet at the fourth power of λ is in the $sl(2) \oplus so(8)$ representation $(0)(0200)$ — the four-index antisymmetric tensors $(0)(0020)$ or $(0)(0002)$ do not occur. So if the structure constants of the 3-algebra defined by $\langle\{x, a\}, \{b, c\}\rangle_{\mathbf{adj}} = \langle x, \llbracket a, b, c \rrbracket \rangle_{\mathbf{R}}$ are antisymmetric, this term vanishes. It was also checked that the commutator of two Ξ -transformations gives a Λ -transformation together with a transformation of the type (9). In this way, one is naturally led to the 3-algebra structure with a minimal amount of input, essentially a “minimal coupling”. I would like to stress that although the pure spinor action contains at most 3-point couplings, the full component action with up to 6-point interactions will arise when unphysical component fields are eliminated.

For the $N = 6$ ABJM models, the results are very similar. Due to lack of time and space, I will not go into details, but refer to ref. [47]. The end result is a weaker condition on the structure constants of the “3-algebra”, which is just the appropriate one [39]. The classification of such algebraic structures was performed in ref. [40]. It is satisfactory that the structure of the pure spinors in both cases give the necessary and sufficient algebraic structure by the vanishing of a single term in the transformation of minimally coupled matter.

In both the $N = 8$ and $N = 6$ theories in $D = 3$, the naïve measure sits at $\lambda^3\theta^3$. In analogy with the ten-dimensional case, we need the number of irreducible constraints on the pure spinors to equal the number of θ 's. Indeed, the constraints, which in both cases

sit in the vector representation of $so(1, 2)$, turn as mentioned out to be irreducible, which is straightforward to check (explicit solutions were given in ref. [47]).

We can write the invariant tensors as

$$\begin{aligned} \varepsilon_{abc}(\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta) \\ = T_{(A_1\alpha_1, A_2\alpha_2, A_3\alpha_3)[B_1\beta_1, B_2\beta_2, B_3\beta_3]} \lambda^{A_1\alpha_1} \lambda^{A_2\alpha_2} \lambda^{A_3\alpha_3} \theta^{B_1\beta_1} \theta^{B_2\beta_2} \theta^{B_3\beta_3} \end{aligned} \quad (14)$$

in the $N = 8$ case, and as a similar expression when $N = 6$. The integration measures are constructed using these invariant tensors in a manner completely analogous to the measure in $D = 10$.

Let us examine the dimensions and ghost numbers of the total measures. The analogies of Table 2 are obtained by simple counting (the $N = 6$ case is included for completeness):

	$N = 8$		$N = 6$	
	gh#	dim	gh#	dim
d^3x	0	-3	0	-3
$[d\theta]$	0	8	0	6
$[d\lambda]$	10	-5	6	-3
$[d\bar{\lambda}]$	-10	5	-6	3
$[dr]$	-3	-5	-3	-3
total	-3	0	-3	0

Table 5. The dimensions and ghost numbers of the $N = 8$ and $N = 6$ measures.

In both cases we get a non-degenerate measure of dimension 0 and ghost number -3, as desired for a conformal theory. Also here, the measures of course have to be regularised in the same way as in ref. [7]. We insert a factor $N = e^{\{\mathcal{Q}, \chi\}}$, where $\chi = -\mu_{A\alpha}\theta^{A\alpha}$ for $N = 8$ (and similarly for $N = 6$).

To conclude, we have presented manifestly supersymmetric formulations of the $N = 8$ BLG models and the $N = 6$ ABJM models. We have also performed a detailed analysis of the pure spinor constraints and provided proper actions based on non-degenerate measures on non-minimal pure spinor spaces. We hope that these formulations may be helpful in the future, *e.g.* for the investigation of quantum properties [49,50] of the models. In order to perform path integrals, one has to gauge fix. Gauge fixing in the pure spinor formalism for the superparticle includes the “ b -ghost”, with the property $\{Q, b\} = \square$. The b -ghost is a composite operator in the pure spinor formalism, since $p^2 = 0$ is not an independent constraint. This operator has singularities at $\lambda = 0$, which need to be regularised. Proposals for resolving this issue and allowing for calculation of string amplitudes at arbitrary loop

level have been made in refs. [51,52], but lead to complicated expressions. It is possible that some simpler approach exists.

I believe that much more is to be learnt from pure spinor superspace formulations, especially of maximally supersymmetric theories. One very interesting example, largely unexplored, is the issue of such formulations of supergravities. I think that the treatment of the scalar multiplet actions in the present work may contain clues to supergravity, both considering the extra gauge invariances and the extra factors of λ in the action. Work is in progress.

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